

# On the Third Gap for Proper Holomorphic Maps between Balls

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## 1 Introduction

The study of proper holomorphic maps between balls in complex spaces of different dimension has attracted much attention in the past many years. Among many problems which people have considered along these lines, we mention at least those questions related to the rationality for maps with minimum boundary regularity, linearity and semi-linearity, gap phenomenons, degree estimates, as well as, various classifications. This paper is devoted to the study of a gap rigidity problem.

To start with, we write  $\mathbb{B}^n$  for the unit ball in the complex space  $\mathbb{C}^n$ . Write  $Prop(\mathbb{B}^n, \mathbb{B}^N)$  for the set of proper holomorphic maps from  $\mathbb{B}^n$  into  $\mathbb{B}^N$ ,  $Prop_k(\mathbb{B}^n, \mathbb{B}^N) \subset Prop(\mathbb{B}^n, \mathbb{B}^N)$  for those maps that are  $C^k$ -smooth up to the boundary, and  $Rat(\mathbb{B}^n, \mathbb{B}^N)$  for the space of proper holomorphic rational maps from  $\mathbb{B}^n$  into  $\mathbb{B}^N$ . We say that  $F$  and  $G \in Prop(\mathbb{B}^n, \mathbb{B}^N)$  are *equivalent* if there are automorphisms  $\sigma \in Aut(\mathbb{B}^n)$  and  $\tau \in Aut(\mathbb{B}^N)$  such that  $F = \tau \circ G \circ \sigma$ .

For a proper holomorphic map  $F \in Prop(\mathbb{B}^n, \mathbb{B}^N)$ , one can always add zero components to  $F$  and then compose it with automorphisms from  $Aut(\mathbb{B}^N)$  to produce other proper holomorphic maps from  $\mathbb{B}^n$  into  $\mathbb{B}^{N'}$  with  $N' > N$ . However, maps obtained in this manner have the same geometric character as that of the original  $F$  and are not regarded as ‘different maps’. A gap rigidity phenomenon for proper holomorphic maps between balls is to ask for which  $N$ , any proper holomorphic map from  $\mathbb{B}^n$  into  $\mathbb{B}^N$ , which has certain boundary regularity, is always equivalent to a map of the form  $(G, 0)$  with  $G$  a proper holomorphic map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N'}$  for  $N' < N$ . Along these lines, in [HJY], the following gap-rigidity conjecture was proposed:

**Conjecture 1.1.** (*Huang-Ji-Yin [HJY]*) *Let  $n \geq 3$  be a positive integer. Write  $K(n)$  for the largest positive integer  $m$  such that  $n > m(m+1)/2$ . For any  $k$  with  $1 \leq k \leq K(n)$ , write*

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\*Supported in part by DMS-1101481

$\mathcal{I}_k$  for the collection of integers  $m'$  such that  $kn < m' < (k+1)n - k(k+1)/2$ . Then, when  $N \in \mathcal{I}_k$  for a certain  $1 \leq k \leq K(n)$ , any proper holomorphic rational map  $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$  is equivalent to a map of the form  $(G, 0)$ , where  $G$  is a proper holomorphic rational map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N'}$  with  $N' = kn < N$ .

Notice that  $\mathcal{I}_1 = \{n+1, \dots, 2n-2\}$  (if  $n \geq 3$ ),  $\mathcal{I}_2 = \{2n+1, 2n+2, \dots, 3n-4\}$  (if  $n \geq 5$ ), and  $\mathcal{I}_3 = \{3n+1, 3n+2, \dots, 4n-7\}$  (if  $n \geq 8$ ). For any  $k \leq K(n)$  as above, we call the non-empty set of positive integers  $\mathcal{I}_k$  the  $k$ -th gap interval for proper holomorphic maps between balls.

It was proved in Theorem 2.8 of [HJY] that for any  $N \notin \mathcal{I}_1 \cup \mathcal{I}_2 \cup \dots \cup \mathcal{I}_{K(n)}$ , there is a map in  $\text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$  that is not equivalent to any map of the form  $(G, 0)$  with  $G \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{N'})$  for  $N' < N$ . In particular the following map in  $\text{Rat}(\mathbb{B}^n, \mathbb{B}^{3n})$  is not equivalent to any map of the form  $(G, 0)$ :

$$F = (z_1, \dots, z_{n-2}, \lambda z_{n-1}, z_n, \sqrt{1-\lambda^2} z_{n-1}(z_1, \dots, z_{n-1}, \mu z_n, \sqrt{1-\mu^2} z_n z)), \quad (1.1)$$

where  $\lambda, \mu \in (0, 1)$ . Hence, the construction in [HJY] shows that one can only expect the gap phenomenon for  $N \in \bigcup_{k=1}^{K(n)} \mathcal{I}_k$  and Conjecture 1.1 actually claims that we do have the gap phenomenon for  $N \in \bigcup_{k=1}^{K(n)} \mathcal{I}_k$ . Here we remark that when  $N \geq \frac{-1+\sqrt{1+4n}}{2}n \approx n^{\frac{3}{2}}$ ,  $N \notin \bigcup_{k=1}^{K(n)} \mathcal{I}_k$ . Earlier in [DL], for  $N \geq n^2 - 2n + 2$ , D'Angelo and Lebl constructed a proper holomorphic monomial map from  $\mathbb{B}^n$  into  $\mathbb{B}^N$ , that is not equivalent to a map of the form  $(G, 0)$  where  $G$  is a proper rational holomorphic map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N'}$  with  $N' < N$ .

On the other hand, making use of the classifications in Faran [Fa], Huang-Ji [HJ], Hamard [Ha] and Huang-Ji-Xu [HJX1] (see [HJY] for more detailed discussions on the history and references on this matter), it is known that, when  $N \in \mathcal{I}_k$  with  $k = 1, 2$ , any rational proper holomorphic map from  $\mathbb{B}^n$  into  $\mathbb{B}^N$  is equivalent to a map of the form  $(G, 0)$ . Conjecture 1.1 thus holds by the work of these people in the case of  $k = 1, 2$ . Notice that if Conjecture 1.1 holds, then there are approximately  $\sqrt{n}$  gap intervals for rational proper maps from  $\mathbb{B}^n$  into balls in complex spaces of larger dimensions.

In this paper, we give a proof of the following result which confirms the above conjecture when  $k = 3$ .

**Theorem 1.2.** *Let  $F \in \text{Prop}_3(\mathbb{B}^n, \mathbb{B}^N)$  with  $3n < N \leq 4n - 7$  and  $n \geq 7$ . Then  $F$  is equivalent to a map of the form  $(G, 0)$  with  $G \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{3n})$ . Namely, any proper rational map  $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$  with  $N \in \mathcal{I}_3$  is equivalent to a map of the form  $(G, 0)$  where  $G$  is a proper rational holomorphic map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N'}$  with  $N' = 3n < N$ .*

We notice that the dimension  $N' = 3n$  in Theorem 1.2 is sharp by the map constructed in (1.1). This example also has the property, as we will make more precisely in the end of

the next section, that it is the largest possible generic map from  $\mathbb{B}^n$  into  $\mathbb{B}^{3n}$ . We also remark that there are already many rational proper holomorphic maps from  $\mathbb{B}^n$  into  $\mathbb{B}^{3n-2}$  that are not equivalent to any proper polynomial maps by the work in Faran-Huang-Ji-Zhang [FHJZ].

For any  $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$  with  $N < \frac{n(n+1)}{2}$ , by the results proved in Huang [Hu2] and Huang-Ji-Xu [HJX2],  $F$  is rational and extends holomorphically across the boundary of  $\mathbb{B}^n$ . Hence, for the proof of Theorem 1.2, we can assume that the map is already rational and extends holomorphically across the boundary. The rationality and holomorphical extendability for proper holomorphic maps into balls has previously studied by many mathematicians. Here, we refer the reader to the work of Forstneric [Fo], Baouendi-Huang-Rothschild [BHR], Mir [Mir], Huang-Ji-Xu [HJX2] and the references therein.

Moreover, for any map  $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$  with  $N < n(n+1)/2$ ,  $F$  has the following partial linearity: For a generic point  $p \in \mathbb{B}^n$ , there is a unique affine subspace  $S_p$  through  $p$  of dimension  $n - \kappa_0 \geq 2$  such that the restriction of  $F$  to  $S_p$  is a linear fractional map. In [Hu2], the first author introduced the concept of geometric rank for  $F \in Prop_2(\mathbb{B}^n, \mathbb{B}^N)$  to measure the degeneracy of the boundary CR second fundamental form of  $F$ . (See Section 2 of the paper for a precise definition). A theorem of [Hu2] states that for  $N < \frac{n(n+1)}{2}$ , the just aforementioned  $\kappa_0$  is precisely the geometric rank of  $F$  and thus can be used as a more intuitive but equivalent definition of the geometric rank of the map. When  $N \leq 4n - 7$ , by a linear algebra formula in [Lemma 3.2, Hu2],  $\kappa_0$  only takes three different value: 0, 1, or 2. The case with  $\kappa_0 = 0$  is trivial for the map is then linear. The classification result of maps with  $\kappa_0 = 1$  in [HJX1] gives, as an easy corollary, Theorem 1.3 for maps with geometric rank 1. Hence, to prove Theorem 1.3, one needs only to consider maps with geometric rank  $\kappa_0 = 2$ . In fact, we will prove in this paper the following slightly more general Theorem 1.3.

**Theorem 1.3.** *Let  $F \in Prop_3(\mathbb{B}^n, \mathbb{B}^N)$  have geometric rank  $\kappa_0 = 2$ . Assume that  $n \geq 7$  and  $3n \leq N \leq 4n - 6$ . Then  $F$  is equivalent to a map of the form  $(G, 0)$  where  $G$  is a rational proper holomorphic map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N'}$  with  $N' = 3n$ .*

One has quite a different phenomenon even for rational proper maps between balls when the codimension gets large. As demonstrated by an early work of Catlin-D'Angelo [CD], one can arbitrarily assign part of the components once one allows the codimension to be sufficiently large. The reader is referred to [CD] for the precise statement and many references therein.

Finally, we mention that the two main ingredients for the proof of our main theorem here include: the normal form obtained in [HJX1] and a basic lemma of the first author [Lemma 3.2, Hu1].

## 2 Notations and Preliminaries

In this section, we set up notation and recall a result established in Huang-Ji-Xu [HJX1].

Write  $\mathbf{H}_n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im}(w) > |z|^2\}$  for the Siegel upper-half space. Similarly, we can define the space  $\text{Rat}(\mathbf{H}_n, \mathbf{H}_N)$ ,  $\text{Prop}_k(\mathbf{H}_n, \mathbf{H}_N)$  and  $\text{Prop}(\mathbf{H}_n, \mathbf{H}_N)$ . Since the Cayley transformation

$$\rho_n : \mathbf{H}_n \rightarrow \mathbb{B}^n, \quad \rho_n(z, w) = \left( \frac{2z}{1-iw}, \frac{1+iw}{1-iw} \right) \quad (2.1)$$

is a biholomorphic mapping between  $\mathbf{H}_n$  and  $\mathbb{B}^n$ , we can identify a map  $F \in \text{Prop}_k(\mathbb{B}^n, \mathbb{B}^N)$  or  $\text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$  with  $\rho_N^{-1} \circ F \circ \rho_n$  in the space  $\text{Prop}_k(\mathbf{H}_n, \mathbf{H}_N)$  or  $\text{Rat}(\mathbf{H}_n, \mathbf{H}_N)$ , respectively.

Parameterize  $\partial\mathbf{H}_n$  by  $(z, \bar{z}, u)$  through the map  $(z, \bar{z}, u) \rightarrow (z, u + i|z|^2)$ . In what follows, we will assign the weight of  $z$  and  $u$  to be 1 and 2, respectively. For a non-negative integer  $m$ , a function  $h(z, \bar{z}, u)$  defined over a small ball  $U$  of 0 in  $\partial\mathbf{H}_n$  is said to be of quantity  $o_{wt}(m)$  if  $\frac{h(tz, t\bar{z}, t^2u)}{|t|^m} \rightarrow 0$  uniformly for  $(z, u)$  on any compact subset of  $U$  as  $t \in \mathbb{R} \rightarrow 0$ . We use the notation  $h^{(k)}$  to denote a polynomial  $h$  which has weighted degree  $k$ . Occasionally, for a holomorphic function (or map)  $H(z, w)$ , we write  $H(z, w) = \sum_{k,l=0}^{\infty} H^{(k,l)}(z)w^l$  with  $H^{(k,l)}(z)$  a polynomial of degree  $k$  in  $z$ .

Let  $F = (f, \phi, g) = (\tilde{f}, g) = (f_1, \dots, f_{n-1}, \phi_1, \dots, \phi_{N-n}, g)$  be a non-constant  $C^2$ -smooth CR map from  $\partial\mathbf{H}_n$  into  $\partial\mathbf{H}_N$  with  $F(0) = 0$ . For each  $p \in M$  close to 0, we write  $\sigma_p^0 \in \text{Aut}(\mathbf{H}_n)$  for the map sending  $(z, w)$  to  $(z + z_0, w + w_0 + 2i\langle z, \bar{z}_0 \rangle)$  and  $\tau_p^F \in \text{Aut}(\mathbf{H}_N)$  by defining

$$\tau_p^F(z^*, w^*) = (z^* - \tilde{f}(z_0, w_0), w^* - \overline{g(z_0, w_0)} - 2i\langle z^*, \overline{\tilde{f}(z_0, w_0)} \rangle).$$

Then  $F$  is equivalent to

$$F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p). \quad (2.2)$$

Notice that  $F_0 = F$  and  $F_p(0) = 0$ . The following is fundamentally important for the understanding of the geometric properties of  $F$ .

**Lemma 2.1** ([§2, Lemma 5.3, Hu99]): Let  $F$  be a  $C^2$ -smooth CR map from  $\partial\mathbf{H}_n$  into  $\partial\mathbf{H}_N$ ,  $2 \leq n \leq N$ . For each  $p \in \partial\mathbf{H}_n$ , there is an automorphism  $\tau_p^{**} \in \text{Aut}_0(\mathbf{H}_N)$  such that  $F_p^{**} := \tau_p^{**} \circ F_p$  satisfies the following normalization:

$$f_p^{**} = z + \frac{i}{2}a_p^{**(1)}(z)w + o_{wt}(3), \quad \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2), \quad g_p^{**} = w + o_{wt}(4), \quad \text{with}$$

$$\langle \bar{z}, a_p^{**(1)}(z) \rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

**Definition 2.2** ([Hu2]) Write  $\mathcal{A}(p) = -2i(\frac{\partial^2(f_p)_{l}^{**}}{\partial z_j \partial w}|_0)_{1 \leq j, l \leq (n-1)}$  in the above lemma. We call the rank of the  $(n-1) \times (n-1)$  matrix  $\mathcal{A}(p)$ , which we denote by  $Rk_F(p)$ , the *geometric rank* of  $F$  at  $p$ .

Define the *geometric rank* of  $F$  to be  $\kappa_0(F) = \max_{p \in \partial \mathbf{H}_n} Rk_F(p)$ . Define the geometric rank of  $F \in \text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$  to be the one for the map  $\rho_N^{-1} \circ F \circ \rho_n \in \text{Prop}_2(\mathbf{H}_n, \mathbf{H}_N)$ . By [Hu2],  $\kappa_0(F)$  depends only on the equivalence class of  $F$  and when  $N < \frac{n(n+1)}{2}$ , the geometric rank  $\kappa_0(F)$  of  $F$  is precisely the  $\kappa_0$  we mentioned in the introduction. In [HJX1], the authors proved the following normalization theorem for maps with degenerate geometric rank, though only part of it is needed later:

**Theorem 2.1.** ([HJX1]) Suppose that  $F \in \text{Rat}(\mathbf{H}_n, \mathbf{H}_N)$  has geometric rank  $1 \leq \kappa_0 \leq n-2$  with  $F(0) = 0$ . Then there are  $\sigma \in \text{Aut}(\mathbf{H}_n)$  and  $\tau \in \text{Aut}(\mathbf{H}_N)$  such that  $\tau \circ F \circ \sigma$  takes the following form, which is still denoted by  $F = (f, \phi, g)$  for convenience of notation:

$$\left\{ \begin{array}{l} f_l = \sum_{j=1}^{\kappa_0} z_j f_{lj}^*(z, w); \quad l \leq \kappa_0 \\ f_j = z_j, \text{ for } \kappa_0 + 1 \leq j \leq n-1; \\ \phi_{lk} = \mu_{lk} z_l z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{lkj}^* \text{ for } (l, k) \in \mathcal{S}_0, \\ g = w; \\ f_{lj}^*(z, w) = \delta_l^j + \frac{i\delta_l^j \mu_l}{2} w + b_{lj}^{(1)}(z)w + O_{wt}(4), \\ \phi_{lkj}^*(z, w) = O_{wt}(2), \quad (l, k) \in \mathcal{S}_1, \\ \phi_{3k} = \sum_{j=1}^{\kappa_0} z_j \phi_{3kj}^* = O_{wt}(3) \text{ for } k = 3, \dots, N - (n + (n-1) + \dots + (n - \kappa_0)) + 2 \end{array} \right. \quad (2.3)$$

Here, for  $1 \leq \kappa_0 \leq n-2$ , we write  $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ , the index set for all components of  $\phi$ , where  $\mathcal{S}_0 = \{(j, l) : 1 \leq j \leq \kappa_0, 1 \leq l \leq n-1, j \leq l\}$  and  $\mathcal{S}_1 = \{(j, l) : j = \kappa_0 + 1, \kappa_0 + 1 \leq l \leq N - n - \frac{(2n - \kappa_0 - 1)\kappa_0}{2}\}$ . Also,  $\mu_{jl} = \sqrt{\mu_j + \mu_l}$  for  $j < l \leq \kappa_0$ ; and  $\mu_{jl} = \sqrt{\mu_j}$  if  $j \leq \kappa_0 < l$  or if  $j = l \leq \kappa_0$ .

Finally, we recall the following lemma of the first author in [Hu1], which will play a fundamental role in our proof:

**Lemma 2.2.** (Huang, Lemma 3.2 [Hu1]) Let  $k$  be a nonnegative integer such that  $1 \leq k \leq n-2$ . Assume that  $a_1, \dots, a_k, b_1, \dots, b_k$  are germs at  $0 \in \mathbb{C}^{n-1}$  of holomorphic functions such that  $a_j(0) = 0, b_j(0) = 0$  and

$$\sum_{i=1}^k a_i(z) \overline{b_i(z)} = A(z, \bar{z})|z|^2, \quad \text{for } j = 1, \dots, n-2, \quad (2.4)$$

where  $A(z, \bar{z})$  is a germ at  $0 \in \mathbb{C}^{n-1}$  of a real analytic function. Then  $A(z, \bar{z}) = \sum_{i=1}^k a_i(z) \overline{b_i(z)} \equiv 0$

### 3 Analysis on the Chern-Moser equation

Suppose now that  $F = (f, \phi, g) \in \text{Rat}(\mathbf{H}_n, \mathbf{H}_N)$  satisfies the normalization as in Theorem 2.1 with  $1 \leq \kappa_0 \leq n - 2$ . Write the codimension part  $\phi$  of the map  $F$  as  $\phi := (\Phi_0, \Phi_1)$  with  $\Phi_0 = (\phi_{\ell k})_{(\ell, k) \in \mathcal{S}_0}$  and  $\Phi_1 = (\phi_{\ell k})_{(\ell, k) \in \mathcal{S}_1}$ . Write  $\Phi_0^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} e_j z_j$  with  $e_j \in \mathbb{C}^{\kappa_0 n - \frac{\kappa_0(\kappa_0+1)}{2}}$ ,  $\xi_j(z) = \overline{e_j} \cdot \Phi_0^{(2,0)}(z)$ ,  $\xi = (\xi_1, \dots, \xi_{\kappa_0})$ . We also write in the following:

$$\begin{aligned} \phi^{(1,1)}(z)w &= \sum e_j^* z_j w, \text{ with } e_j^* = (e_j, \hat{e}_j), \\ H &= \sum_{(i_1, \dots, i_{n-1}, i_n)} H^{(i_1, \dots, i_n)} z_1^{i_1} \dots z_{n-1}^{i_{n-1}} w^{i_n} = \sum_{k,j=0}^{\infty} H^{(k,j)}(z) w^j \text{ for } H = f \text{ or } \phi. \end{aligned}$$

Here  $H^{(k,j)}(z)$  is a polynomial of degree  $k$  in  $z$ .

In this section, we demonstrate our basic idea of the proof through an easier case. We proceed with the following lemma, that will be used later:

**Lemma 3.1.** *Let  $(\Gamma_j^{[h]}(z))_{1 \leq j \leq \kappa_0, h=1,2}$  be some holomorphic functions of  $z$ . Let  $\mu_{jl}$   $\mu_j$  be as defined in Theorem 2.1. Suppose that for  $h = 1, 2$ ,  $(\Lambda_{j\ell}^{[h]})_{(j,\ell) \in \mathcal{S}_0}$  are defined as follows:*

1.  $\mu_{j\ell} \Lambda_{j\ell}^{[h]}(z) = 2i(z_j \Gamma_\ell^{[h]} + z_\ell \Gamma_j^{[h]}), \quad j < \ell \leq \kappa_0,$
2.  $\mu_{jj} \Lambda_{jj}^{[h]}(z) = 2iz_j \Gamma_j^{[h]}(z), \quad j \leq \kappa_0,$
3.  $\mu_{j\ell} \Lambda_{j\ell}^{[h]} = 2iz_\ell \Gamma_j^{[h]}(z), \quad j \leq \kappa_0 < \ell.$

Then we have

$$\begin{aligned} \sum_{(j,\ell) \in \mathcal{S}_0} \overline{\Lambda_{j\ell}^{[1]}} \Lambda_{j\ell}^{[2]} &= 4|z|^2 \left( \sum_{j \leq \kappa_0} \frac{1}{\mu_j} \overline{\Gamma_j^{[1]}} \Gamma_j^{[2]} \right) - \frac{4}{\mu_1 \mu_2 (\mu_1 + \mu_2)} \sum_{j < \ell \leq \kappa_0} (\mu_j \overline{z_j} \overline{\Gamma_\ell^{[1]}} - \mu_\ell \overline{z_\ell} \overline{\Gamma_j^{[1]}}) \\ &\quad \cdot (\mu_j z_j \Gamma_\ell^{[2]} - \mu_\ell z_\ell \Gamma_j^{[2]}). \end{aligned} \tag{3.1}$$

*Proof.* Making use of the formulas between  $\mu_{j\ell}$  and  $\mu_j, \mu_\ell$  in Theorem 2.1, we get, from a

straightforward computation, the following:

$$\begin{aligned}
\frac{1}{4} \sum_{(j,\ell) \in \mathcal{S}_0} \overline{\Lambda_{j\ell}^{[1]}} \Lambda_{j\ell}^{[2]} &= \sum_{1 \leq j \leq \kappa_0} \frac{|z_j|^2}{\mu_j} \overline{\Gamma_j^{[1]}} \Gamma_j^{[2]} + \sum_{j \leq \kappa_0 < \ell} \frac{|z_\ell|^2}{\mu_j} \overline{\Gamma_j^{[1]}} \Gamma_j^{[2]} \\
&\quad + \sum_{j < \ell \leq \kappa_0} \frac{1}{\mu_j + \mu_\ell} \overline{(z_j \Gamma_\ell^{[1]} + z_\ell \Gamma_j^{[1]})} \cdot (z_j \Gamma_\ell^{[2]} + z_\ell \Gamma_j^{[2]}) \\
&= \left( \sum_{j \leq \kappa_0} \frac{1}{\mu_j} \overline{\Gamma_j^{[1]}} \Gamma_j^{[2]} \right) |z|^2 - \sum_{\ell \leq \kappa_0, \ell \neq j \leq \kappa_0} \frac{1}{\mu_j} |z_\ell|^2 \overline{\Gamma_j^{[1]}} \Gamma_j^{[2]} \\
&\quad + \sum_{j < \ell \leq \kappa_0} \frac{1}{\mu_j + \mu_\ell} \overline{(z_j \Gamma_\ell^{[1]} + z_\ell \Gamma_j^{[1]})} \cdot (z_j \Gamma_\ell^{[2]} + z_\ell \Gamma_j^{[2]}).
\end{aligned}$$

Now the lemma follows from the following elementary identity:

$$\begin{aligned}
&\frac{\mu_j}{\mu_\ell} |z_j|^2 \overline{\Gamma_\ell^{[1]}} \Gamma_\ell^{[2]} + \frac{\mu_\ell}{\mu_j} |z_\ell|^2 \overline{\Gamma_j^{[1]}} \Gamma_j^{[2]} - 2 \operatorname{Re}(z_j \Gamma_\ell^{[2]} \overline{\Gamma_j^{[1]}} \overline{z_\ell}) \\
&= \frac{1}{\mu_j \mu_\ell} (\mu_j \overline{z_j} \overline{\Gamma_\ell^{[1]}} - \mu_\ell \overline{z_\ell} \overline{\Gamma_j^{[1]}}) \cdot (\mu_j z_j \Gamma_\ell^{[2]} - \mu_\ell z_\ell \Gamma_j^{[2]}).
\end{aligned}$$

□

Next we derive the following formula:

**Lemma 3.2.**

$$\frac{1}{4} |\Phi_0^{(3,0)}(z)|^2 = \left( \sum_{j \leq \kappa_0} \frac{1}{\mu_j} |\xi_j(z)|^2 \right) |z|^2 - \sum_{j < \ell \leq \kappa_0} \frac{1}{\mu_j + \mu_\ell} \left| \sqrt{\frac{\mu_j}{\mu_\ell}} z_j \xi_\ell - \sqrt{\frac{\mu_\ell}{\mu_j}} z_\ell \xi_j \right|^2. \quad (3.2)$$

*Proof.* Since

$$-\operatorname{Im}(g) + |f|^2 + |\phi|^2 = 0 \quad \text{over} \quad \operatorname{Im}(w) = |z|^2, \quad (3.3)$$

we can consider terms of weighted degree 5 to get

$$z \overline{f^{(4)}} + \overline{z} f^{(4)} + \Phi_0^{(2)} \overline{\Phi_0^{(3)}} + \Phi_0^{(3)} \overline{\Phi_0^{(2)}} = 0, \quad \text{over} \quad \operatorname{Im}(w) = |z|^2, \quad \text{or} \quad (3.4)$$

$$\begin{aligned}
&\overline{z f^{(2,1)}(z)(u + i|z|^2)} + \overline{z} f^{(2,1)}(z)(u + i|z|^2) + \Phi_0^{(2)}(z) \overline{\left( \Phi_0^{(3,0)}(z) + \left( \sum e_j z_j \right) w \right)} \\
&\quad + \left( \Phi_0^{(3,0)}(z) + \left( \sum e_j z_j \right) w \right) \overline{\Phi_0^{(2)}(z)} \equiv 0.
\end{aligned}$$

Here, we know  $f^{(4)}(z, w) = f^{(2,1)}(z)w$  by the above mentioned normalization. Collecting terms of the form  $\bar{z}^\alpha z^\beta u$  with  $|\alpha| = 1$ ,  $|\beta| = 2$ , we get

$$\bar{z}f^{(2,1)}(z) + \Phi_0^{(2)}(z)\overline{\sum e_j z_j} = 0, \quad \text{or}$$

$$\bar{z}f^{(2,1)}(z) = -\overline{(z_1, \dots, z_{\kappa_0})} \cdot \xi(z). \quad (3.5)$$

Collecting terms of the form  $z^\alpha \bar{z}^\beta$  with  $|\alpha| = 3$  and  $|\beta| = 2$ , we get

$$i\bar{z}f^{(2,1)}(z)|z|^2 + \overline{\Phi_0^{(2)}(z)\Phi_0^{(3,0)}(z)} + \Phi_0^{(2)}(z)\overline{\sum_{j=1}^{\kappa_0} e_j z_j (i|z|^2)} \equiv 0. \quad (3.6)$$

We thus get

$$\overline{\Phi_0^{(2)}(z)\Phi_0^{(3,0)}(z)} = 2i\overline{(z_1, \dots, z_{\kappa_0})} \cdot \xi(z)|z|^2. \quad (3.7)$$

Equivalently, we have

1.  $\mu_{j\ell}\phi_{j\ell}^{(3,0)}(z) = 2i(z_j\xi_\ell + z_\ell\xi_j), \quad j < \ell \leq \kappa_0,$
2.  $\mu_{jj}\phi_{jj}^{(3,0)}(z) = 2iz_j\xi_j(z), \quad j \leq \kappa_0,$
3.  $\mu_{j\ell}\phi_{j\ell}^{(3,0)} = 2iz_\ell\xi_j(z), \quad j \leq \kappa_0 < \ell.$

Now Lemma 3.2 follows from Lemma 3.1. □

**Lemma 3.3.**  $|\phi^{(3,0)}|^2 = A(z, \bar{z})|z|^2$  with  $A(z, \bar{z})$  a real analytic polynomial in  $(z, \bar{z})$ .

*Proof:* Collecting terms of weighted degree 6 in (3.3), we get

$$z\overline{f^{(5)}} + \bar{z}f^{(5)} + \Phi_0^{(2)}\overline{\Phi_0^{(4)}} + \overline{\Phi_0^{(4)}}\Phi_0^{(2)} + |\phi^{(3)}|^2 + |f^{(3)}(z, w)|^2 = 0.$$

Collecting terms of the form  $z^\alpha \bar{z}^\beta$  with  $|\alpha| = |\beta| = 3$  and applying the normalization for  $F$ , we easily see the proof (cf., (4.14) below). □

Notice that  $|\phi^{(3,0)}|^2 = |\Phi_0^{(3,0)}|^2 + |\Phi_1^{(3,0)}|^2$  and there are  $\frac{\kappa_0(\kappa_0+1)}{2} - \kappa_0$  negative terms in the right hand side of (3.2). Also there are  $(N - (\kappa_0 + 1)n + \frac{\kappa_0(\kappa_0+1)}{2})$  components in  $\Phi_1$ . Applying Lemma 2.2 and the D'Angelo Lemma (after a rotation if needed), we immediately get the following:

**Corollary 3.4.** *If  $N \leq (\kappa_0 + 2)n - \kappa_0(\kappa_0 + 1) + \kappa_0 - 2$ , then*

$$\Phi_1^{(3,0)}(z) = \left( \frac{2}{\sqrt{\mu_j + \mu_l}} \left( \sqrt{\frac{\mu_j}{\mu_\ell}} z_j \xi_\ell - \sqrt{\frac{\mu_\ell}{\mu_j}} z_\ell \xi_j \right), 0' \right)_{1 \leq j < l \leq \kappa_0}, \quad |\phi^{(3,0)}(z)|^2 = 4 \left( \sum_{j \leq \kappa_0} \frac{1}{\mu_j} |\xi_j(z)|^2 \right) |z|^2.$$



## 4 Further applications of the partial linearity and the Chern-Moser equation

We assume in this section that  $F = (f, \phi, g) \in \text{Rat}(\mathbf{H}_n, \mathbf{H}_N)$  satisfies the normalization as in Theorem 2.1 with  $\kappa_0 = 2$ . Moreover, by what is done in the last section, we assume that  $\Phi_1^{(3,0)}$  has been normalized to take the form as in Corollary 3.4. Namely, the only possible non-zero element in  $\Phi_1^{(3,0)}$  is  $\phi_{33}^{(3,0)}$ .

In this section, we prove the following result, which will be crucial for our proof of Theorem 1.3:

**Proposition 4.1.** *Assume  $F$  is as in Theorem 2.1 with  $\kappa_0 = 2$  and  $N \leq 4n - 6$ . Also, assume  $\Phi_0^{(3,0)}$  is normalized as in Corollary 3.4. Then the following holds:*

(1):  $\Phi_1^{(4,0)}(z) = (\phi_{33}^{(4,0)}(z), 0, \dots, 0)$ , where

$$\phi_{33}^{(4,0)}(z) = \frac{2}{\sqrt{\mu_1 + \mu_2}} \left( \sqrt{\frac{\mu_1}{\mu_2}} z_1 \eta_2^* - \sqrt{\frac{\mu_2}{\mu_1}} z_2 \eta_1^* \right), \quad \eta_1^* = \phi^{(3,0)} \cdot \overline{e_1^*}, \quad \eta_2^* = \phi^{(3,0)} \cdot \overline{e_2^*}.$$

(2):  $D_z^\alpha \Phi_1^{(2,1)} \in \text{span}\{(1, 0, \dots, 0), \hat{e}_1, \hat{e}_2\}$  for  $|\alpha| = 2$ .

(3):  $D_z^\alpha \Phi_1^{(1,2)} \in \text{span}\{\hat{e}_1, \hat{e}_2\}$  for  $|\alpha| = 1$ .

Here  $\hat{e}_1, \hat{e}_2, e_1^*, e_2^*$ , are defined as at the beginning of the last section, and  $D$  is the regular differential operator.

This section is devoted to the proof of the above theorem:

Notice that  $g = w$ . By a theorem of the first author in [Hu2], we can assume that for any  $\epsilon = (\epsilon_1, \epsilon_2) \in \mathbb{C}^2 \approx 0$ , there is a unique affine subspace  $L_\epsilon$  of codimension two defined by equations of the form:

$$z_1 = \sum_{i=3}^{n-1} a_i(\epsilon) z_i + a_n(\epsilon) w + \epsilon_1, \quad z_2 = \sum_{i=3}^{n-1} b_i(\epsilon) z_i + b_n(\epsilon) w + \epsilon_2, \quad a_i(0) = b_i(0) = 0 \quad (4.1)$$

such that  $F$  is a linear map on  $L_\epsilon$ . Here  $a_j, b_j$  are holomorphic functions in  $\epsilon$  near 0. Hence we have

$$\frac{\partial^2 H}{\partial w^2} \Big|_{L_\epsilon} = 0 \text{ for } H = f \text{ or } \phi.$$

Namely, for  $H(L_\epsilon) = H\left(\sum_{i=3}^{n-1} a_i(\epsilon) z_i + a_n(\epsilon) w + \epsilon_1, \sum_{i=3}^{n-1} b_i(\epsilon) z_i + b_n(\epsilon) w + \epsilon_2, z_3, \dots, z_{n-1}, w\right)$ , we have

$$\begin{aligned} 0 &= \frac{\partial^2 H(L_\epsilon)}{\partial w^2} \Big|_{(\epsilon_1, \epsilon_2)} \\ &= \left( \frac{\partial^2 H}{\partial^2 z_1^2} a_n^2 + \frac{\partial^2 H}{\partial^2 z_2^2} b_n^2 + 2 \frac{\partial^2 H}{\partial z_1 \partial z_2} a_n b_n + 2 \frac{\partial^2 H}{\partial z_1 \partial w} a_n + 2 \frac{\partial^2 H}{\partial z_2 \partial w} b_n + \frac{\partial^2 H}{\partial w^2} \right) \Big|_{(\epsilon_1, \epsilon_2, 0, \dots, 0)}. \end{aligned} \quad (4.2)$$

Let  $a_n^{(1)}(\epsilon)$  and  $b_n^{(1)}(\epsilon)$  be the linear parts in  $a_n$  and  $b_n$ , respectively. Set  $H = f_1, f_2$  and  $\phi$  in (4.2), respectively. We then get

$$\begin{aligned} \frac{i}{2}\mu_1 a_n^{(1)}(\epsilon) + f_1^{(1,2)}(\epsilon, 0, \dots, 0) &= 0, \quad \frac{i}{2}\mu_2 b_n^{(1)}(\epsilon) + f_2^{(1,2)}(\epsilon, 0, \dots, 0) = 0, \\ \phi^{(1,2)}(\epsilon, 0, \dots, 0) + e_1^* a_n^{(1)}(\epsilon) + e_2^* b_n^{(1)}(\epsilon) &= 0. \end{aligned} \quad (4.3)$$

Notice that by Theorem 2.1,  $F^{(1,m)}(z)$  depends only on  $(z_1, z_2)$  for any  $m$ . It then follows:

$$\phi^{(1,2)} = -e_1^* a_n^{(1)} - e_2^* b_n^{(1)} = -\frac{2i}{\mu_1} f_1^{(1,2)} e_1^* - \frac{2i}{\mu_2} f_2^{(1,2)} e_2^*. \quad (4.4)$$

This proves Proposition 4.1 (3). Moreover, we obtain

$$\begin{aligned} \overline{\Phi_0^{(1,2)}} \cdot \Phi_0^{(2,0)} &= \frac{2i}{\mu_1} \overline{f_1^{(1,2)}} \overline{e_1} \cdot \Phi_0^{(2,0)} + \frac{2i}{\mu_2} \overline{f_2^{(1,2)}} \overline{e_2} \cdot \Phi_0^{(2,0)} \\ &= \frac{2i}{\mu_1} (\overline{f_1^{(I_1+2I_n)}} \overline{z_1} + \overline{f_1^{(I_2+2I_n)}} \overline{z_2}) \xi_1 + \frac{2i}{\mu_2} (\overline{f_2^{(I_1+2I_n)}} \overline{z_1} + \overline{f_2^{(I_2+2I_n)}} \overline{z_2}) \xi_2. \end{aligned} \quad (4.5)$$

Here and in what follows, write  $I_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n$ , where the non-zero element 1 is in the  $j^{th}$ -position. From (4.5), we also have

$$\begin{aligned} \overline{\Phi_0^{(I_1+2I_n)}} \cdot \Phi_0^{(2,0)} &= 2i \left( \frac{\xi_1}{\mu_1} \overline{f_1^{(I_1+2I_n)}} + \frac{\xi_2}{\mu_2} \overline{f_2^{(I_1+2I_n)}} \right) \\ \overline{\Phi_0^{(I_2+2I_n)}} \cdot \Phi_0^{(2,0)} &= 2i \left( \frac{\xi_1}{\mu_1} \overline{f_1^{(I_2+2I_n)}} + \frac{\xi_2}{\mu_2} \overline{f_2^{(I_2+2I_n)}} \right), \end{aligned} \quad (4.6)$$

and the following:

$$\begin{aligned} &2i \left( \frac{\overline{\xi_1}}{\mu_1} \overline{\Phi_0^{(I_1+2I_n)}} \cdot \Phi_0^{(2,0)} + \frac{\overline{\xi_2}}{\mu_2} \overline{\Phi_0^{(I_2+2I_n)}} \cdot \Phi_0^{(2,0)} \right) \\ &= \frac{-4\overline{\xi_1}}{\mu_1} \cdot \left( \frac{\xi_1}{\mu_1} \overline{f_1^{(I_1+2I_n)}} + \frac{\xi_2}{\mu_2} \overline{f_2^{(I_1+2I_n)}} \right) + \frac{-4\overline{\xi_2}}{\mu_2} \cdot \left( \frac{\xi_1}{\mu_1} \overline{f_1^{(I_2+2I_n)}} + \frac{\xi_2}{\mu_2} \overline{f_2^{(I_2+2I_n)}} \right) \\ &= -4 \frac{\overline{\xi_1}}{\mu_1} \left( \overline{f_1^{(I_1+2I_n)}} \frac{\overline{\xi_1}}{\mu_1} + \overline{f_1^{(I_2+2I_n)}} \frac{\overline{\xi_2}}{\mu_2} \right) - 4 \frac{\overline{\xi_2}}{\mu_2} \left( \overline{f_2^{(I_1+2I_n)}} \frac{\overline{\xi_1}}{\mu_1} + \overline{f_2^{(I_2+2I_n)}} \frac{\overline{\xi_2}}{\mu_2} \right). \end{aligned} \quad (4.7)$$

Considering terms of weighted degree 6 in the basic equation (3.3), we get

$$2\operatorname{Re} \left\{ \overline{z} f^{(5)} + \overline{\Phi_0^{(2)}} \Phi_0^{(4)} \right\} + |f^{(3)}|^2 + |\phi^{(3)}|^2 = 0. \quad (4.8)$$

Namely, we have

$$2\operatorname{Re}\left\{\overline{z}\left(f^{(3,1)}(z)(u+i|z|^2)+f^{(1,2)}(z)(u+i|z|^2)^2\right)+\overline{\Phi_0^{(2,0)}(z)}\left(\Phi_0^{(4,0)}(z)+\Phi_0^{(2,1)}(z)\right.\right. \\ \left.\left.\cdot(u+i|z|^2)\right)\right\}+|f^{(1,1)}(z)(u+i|z|^2)|^2+|\phi^{(3,0)}(z)+\phi^{(1,1)}(z)(u+i|z|^2)|^2=0. \quad (4.9)$$

Collecting terms of the form  $z^\alpha\overline{z}^\beta u^2$  with  $|\alpha|=1, |\beta|=1$ , we get

$$2\operatorname{Re}(\overline{z}f^{(1,2)}(z))+|f^{(1,1)}(z)|^2+|\phi^{(1,1)}(z)|^2=0. \quad (4.10)$$

Collecting terms of the form  $z^\alpha\overline{z}^\beta u$  with  $|\alpha|=3, |\beta|=1$ , we get

$$\overline{z}f^{(3,1)}(z)+\phi^{(3,0)}(z)\cdot\overline{\phi^{(1,1)}(z)}=0. \quad (4.11)$$

Collecting terms of the form  $z^\alpha\overline{z}^\beta u$  with  $|\alpha|=2, |\beta|=2$ , we get

$$2\operatorname{Re}\left(2i\overline{z}f^{(1,2)}(z)|z|^2+\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(2,1)}(z)\right)=0. \quad (4.12)$$

Collecting terms of the form  $z^\alpha\overline{z}^\beta$  with  $|\alpha|=4, |\beta|=2$ , we get

$$i|z|^2\overline{z}f^{(3,1)}(z)+\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(4,0)}(z)-i|z|^2\overline{\phi^{(1,1)}(z)}\cdot\phi^{(3,0)}(z)=0. \quad (4.13)$$

Collecting terms of the form  $z^\alpha\overline{z}^\beta$  with  $|\alpha|=3, |\beta|=3$ , we get

$$2\operatorname{Re}\left(-\overline{z}f^{(1,2)}(z)|z|^4+i|z|^2\overline{\Phi_0^{(2,0)}}\cdot\Phi_0^{(2,1)}\right) \\ +|z|^4\cdot|f^{(1,1)}(z)|^2+|\phi^{(3,0)}(z)|^2+|z|^4\cdot|\phi^{(1,1)}(z)|^2=0. \quad (4.14)$$

Combining (4.11) with (4.13), we get

$$\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(4,0)}(z)=2i|z|^2\overline{\phi^{(1,1)}(z)}\cdot\phi^{(3,0)}(z). \quad (4.15)$$

Substituting (4.10) into (4.14), we get

$$2\operatorname{Re}\left(-2\overline{z}f^{(1,2)}(z)|z|^2+i\overline{\Phi_0^{(2,0)}}\cdot\Phi_0^{(2,1)}(z)\right)|z|^2+|\phi^{(3,0)}(z)|^2=0. \quad (4.16)$$

Combining (4.12) with (4.16), we get

$$2\left(-2\overline{z}f^{(1,2)}(z)|z|^2+i\overline{\Phi_0^{(2,0)}}\cdot\Phi_0^{(2,1)}(z)\right)|z|^2+|\phi^{(3,0)}(z)|^2=0. \quad (4.17)$$

Recall that in Corollary 3.4, we have obtained

$$|\phi^{(3,0)}|^2 = 4|z|^2 \left( \frac{|\xi_1|^2}{\mu_1} + \frac{|\xi_2|^2}{\mu_2} \right), \quad (4.18)$$

Hence we have

$$2 \left( -2\bar{z}f^{(1,2)}(z)|z|^2 + i\overline{\Phi_0^{(2,0)}} \cdot \Phi_0^{(2,1)}(z) \right) + 4 \left( \frac{|\xi_1|^2}{\mu_1} + \frac{|\xi_2|^2}{\mu_2} \right) = 0. \quad (4.19)$$

Notice that  $|\xi_i|^2 = \bar{\xi}_i \xi_i = \xi_i e_i \cdot \overline{\Phi_0^{(2,0)}}$ . Set

$$\tilde{\phi}^{(2,1)}(z) = \phi^{(2,1)}(z) - 2i \sum_{j=1}^2 \frac{\xi_j}{\mu_j} e_j^*, \quad \tilde{\Phi}_0^{(2,1)}(z) = \Phi_0^{(2,1)}(z) - 2i \sum_{j=1}^2 \frac{\xi_j}{\mu_j} e_j. \quad (4.20)$$

Then we have

$$\overline{\Phi_0^{(2,0)}}(z) \cdot \tilde{\Phi}_0^{(2,1)}(z) = -2i|z|^2 \bar{z} \cdot f^{(1,2)}(z). \quad (4.21)$$

Thus we get

$$\begin{aligned} \tilde{\Phi}_{11}^{(2,1)}(z) &= \frac{-2i}{\sqrt{\mu_1}} z_1 f_1^{(1,2)}, \quad \tilde{\Phi}_{12}^{(2,1)}(z) = \frac{-2i}{\sqrt{\mu_1 + \mu_2}} (z_1 f_2^{(1,2)} + z_2 f_1^{(1,2)}), \quad \tilde{\Phi}_{22}^{(2,1)}(z) = \frac{-2i}{\sqrt{\mu_2}} z_2 f_2^{(1,2)}, \\ \tilde{\Phi}_{1j}^{(2,1)}(z) &= \frac{-2i}{\sqrt{\mu_1}} z_j f_1^{(1,2)}, \quad \tilde{\Phi}_{2j}^{(2,1)}(z) = \frac{-2i}{\sqrt{\mu_2}} z_j f_2^{(1,2)} \text{ with } j \geq 3. \end{aligned} \quad (4.22)$$

Making use of Lemma 3.1, we get

$$|\tilde{\Phi}_0^{(2,1)}(z)|^2 = 4|z|^2 \sum_{j=1}^2 \frac{1}{\mu_j} |f_j^{(1,2)}(z)|^2 - \frac{4}{\mu_1 \mu_2 (\mu_1 + \mu_2)} |\mu_1 z_1 f_2^{(1,2)}(z) - \mu_2 z_2 f_1^{(1,2)}(z)|^2. \quad (4.23)$$

and

$$\begin{aligned} \overline{\tilde{\Phi}_0^{(2,1)}}(z) \Phi_0^{(3,0)}(z) &= -4|z|^2 \sum_{j=1}^2 \frac{1}{\mu_j} \overline{f_j^{(1,2)}(z)} \xi_j \\ &\quad + \frac{4}{\mu_1 \mu_2 (\mu_1 + \mu_2)} \left( \mu_1 z_1 f_2^{(1,2)}(z) - \mu_2 z_2 f_1^{(1,2)}(z) \right) \cdot \left( \mu_1 z_1 \xi_2 - \mu_2 z_2 \xi_1 \right). \end{aligned} \quad (4.24)$$

Notice that if we replace  $z_1, z_2$  by  $\frac{\xi_1}{\mu_1}, \frac{\xi_2}{\mu_2}$ , respectively, in (4.10), we get

$$\begin{aligned} & 2\operatorname{Re}\left\{\frac{\bar{\xi}_1}{\mu_1}\left(f_1^{(I_1+2I_n)}\frac{\xi_1}{\mu_1}+f_1^{(I_2+2I_n)}\frac{\xi_2}{\mu_2}\right)+\frac{\bar{\xi}_2}{\mu_2}\left(f_2^{(I_1+2I_n)}\frac{\xi_1}{\mu_1}+f_2^{(I_2+2I_n)}\frac{\xi_2}{\mu_2}\right)\right\} \\ & +\frac{1}{4}(|\xi_1|^2+|\xi_2|^2)+\left|\frac{\xi_1}{\mu_1}e_1^*+\frac{\xi_2}{\mu_2}e_2^*\right|^2=0. \end{aligned} \quad (4.25)$$

Here we have used  $f_j^{(1,1)}(z)=\frac{i}{2}\mu_j z_j$  for  $j=1,2$ . Combing this with (4.7), we get

$$\begin{aligned} & -2\operatorname{Re}\left\{2i\left(\frac{\bar{\xi}_1}{\mu_1}\overline{\Phi_0^{(I_1+2I_n)}}+\frac{\bar{\xi}_2}{\mu_2}\overline{\Phi_0^{(I_2+2I_n)}}\right)\cdot\Phi_0^{(2,0)}\right\} \\ & +(|\xi_1|^2+|\xi_2|^2)+4\left|\frac{\xi_1}{\mu_1}e_1^*+\frac{\xi_2}{\mu_2}e_2^*\right|^2=0. \end{aligned} \quad (4.26)$$

Considering terms of weighted degree 7 in the basic equation (3.3), we get

$$2\operatorname{Re}\left\{\bar{z}f^{(6)}+\overline{f^{(3)}}f^{(4)}+\overline{\Phi_0^{(2)}}\Phi_0^{(5)}+\overline{\phi^{(3)}}\phi^{(4)}\right\}=0. \quad (4.27)$$

Namely, we have

$$\begin{aligned} & 2\operatorname{Re}\left\{\bar{z}\left(f^{(4,1)}(z)(u+i|z|^2)+f^{(2,2)}(z)(u+i|z|^2)^2\right)+\overline{f^{(1,1)}(z)(u+i|z|^2)}\cdot f^{(2,1)}(z)\right. \\ & \cdot (u+i|z|^2)+\overline{\Phi_0^{(2,0)}(z)}\left(\Phi_0^{(5,0)}(z)+\Phi_0^{(3,1)}(z)(u+i|z|^2)+\Phi_0^{(1,2)}(z)(u+i|z|^2)^2\right) \\ & \left.+\overline{(\phi^{(3,0)}(z)+\phi^{(1,1)}(z)(u+i|z|^2))}\cdot(\phi^{(4,0)}(z)+\phi^{(2,1)}(z)(u+i|z|^2))\right\}=0. \end{aligned} \quad (4.28)$$

Collecting terms of the form  $z^\alpha\bar{z}^\beta u^2$  with  $|\alpha|=2, |\beta|=1$ , we get

$$\bar{z}f^{(2,2)}(z)+\overline{f^{(1,1)}(z)}\cdot f^{(2,1)}(z)+\overline{\Phi_0^{(1,2)}(z)}\cdot\Phi_0^{(2,0)}(z)+\overline{\phi^{(1,1)}(z)}\cdot\phi^{(2,1)}(z)=0. \quad (4.29)$$

Collecting terms of the form  $z^\alpha\bar{z}^\beta u$  with  $|\alpha|=3, |\beta|=2$ , we get

$$2i\bar{z}|z|^2f^{(2,2)}+\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z)-2i|z|^2\overline{\Phi_0^{(1,2)}(z)}\cdot\Phi_0^{(2,0)}(z)+\overline{\phi^{(2,1)}(z)}\cdot\phi^{(3,0)}(z)=0. \quad (4.30)$$

Collecting terms of the form  $z^\alpha\bar{z}^\beta$  with  $|\alpha|=4, |\beta|=3$ , we get

$$\begin{aligned} & -\bar{z}f^{(2,2)}(z)|z|^4+\overline{f^{(1,1)}(z)}\cdot f^{(2,1)}(z)|z|^4+i|z|^2\overline{\Phi_0^{(2,0)}(z)}\cdot\Phi_0^{(3,1)}(z)-|z|^4\overline{\Phi_0^{(1,2)}(z)}\cdot\Phi_0^{(2,0)}(z) \\ & +\overline{\phi^{(3,0)}(z)}\cdot\phi^{(4,0)}(z)-i|z|^2\overline{\phi^{(2,1)}(z)}\cdot\phi^{(3,0)}(z)+|z|^4\overline{\phi^{(1,1)}(z)}\cdot\phi^{(2,1)}(z)=0. \end{aligned} \quad (4.31)$$

By calculating (4.31)– $|z|^4(4.29)$ , we get

$$\begin{aligned} & -2\bar{z}f^{(2,2)}(z)|z|^4 + i|z|^2\overline{\Phi_0^{(2,0)}(z)}\Phi_0^{(3,1)}(z) - 2|z|^4\overline{\Phi_0^{(1,2)}(z)} \cdot \Phi_0^{(2,0)}(z) \\ & + \overline{\phi^{(3,0)}(z)} \cdot \phi^{(4,0)}(z) - i|z|^2\overline{\phi^{(2,1)}(z)} \cdot \phi^{(3,0)}(z) = 0. \end{aligned} \quad (4.32)$$

By calculating (4.32)– $i|z|^2(4.30)$ , we get

$$\overline{\phi^{(3,0)}(z)} \cdot \phi^{(4,0)}(z) = 4|z|^4\overline{\Phi_0^{(1,2)}(z)} \cdot \Phi_0^{(2,0)}(z) + 2i|z|^2\overline{\phi^{(2,1)}(z)} \cdot \phi^{(3,0)}(z). \quad (4.33)$$

Combining this with (4.30), we get

$$\overline{\phi^{(3,0)}(z)} \cdot \phi^{(4,0)}(z) = -2i|z|^2 \left( 2i|z|^2\bar{z}f^{(2,2)}(z) + \overline{\Phi_0^{(2,0)}(z)} \cdot \Phi_0^{(3,1)}(z) \right). \quad (4.34)$$

By (4.15), we have

$$\begin{aligned} \mu_{11} \cdot \Phi_{11}^{(4,0)} &= 2iz_1\phi^{(3,0)} \cdot \bar{e}_1^*, \\ \mu_{12} \cdot \Phi_{12}^{(4,0)} &= 2iz_1\phi^{(3,0)} \cdot \bar{e}_2^* + 2iz_2\phi^{(3,0)} \cdot \bar{e}_1^*, \\ \mu_{22} \cdot \Phi_{22}^{(4,0)} &= 2iz_2\phi^{(3,0)} \cdot \bar{e}_2^*, \\ \mu_{1j} \cdot \Phi_{1j}^{(4,0)} &= 2iz_j\phi^{(3,0)} \cdot \bar{e}_1^*, \quad j \geq 3 \\ \mu_{2j} \cdot \Phi_{2j}^{(4,0)} &= 2iz_j\phi^{(3,0)} \cdot \bar{e}_2^*, \quad j \geq 3. \end{aligned} \quad (4.35)$$

Write

$$\eta_1^* = \phi^{(3,0)} \cdot \bar{e}_1^*, \quad \eta_2^* = \phi^{(3,0)} \cdot \bar{e}_2^*; \quad \eta_1 = \Phi_0^{(3,0)} \cdot \bar{e}_1, \quad \eta_2 = \Phi_0^{(3,0)} \cdot \bar{e}_2.$$

Making use of Lemma 3.1, we get

$$\begin{aligned} \overline{\Phi_0^{(3,0)}}\Phi_0^{(4,0)} &= 4|z|^2 \left( \frac{\bar{\xi}_1\eta_1^*}{\mu_1} + \frac{\bar{\xi}_2\eta_2^*}{\mu_2} \right) \\ &\quad - \frac{4}{\mu_1 + \mu_2} \left( \sqrt{\frac{\mu_2}{\mu_1}}\bar{z}_2\bar{\xi}_1 - \sqrt{\frac{\mu_1}{\mu_2}}\bar{z}_1\bar{\xi}_2 \right) \cdot \left( \sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^* \right). \end{aligned} \quad (4.36)$$

Combining (4.34) with (4.36) and making use of Lemma 2.2, we get

$$\overline{\phi_{33}^{(3,0)}}\phi_{33}^{(4,0)} = \frac{4}{\mu_1 + \mu_2} \left( \sqrt{\frac{\mu_2}{\mu_1}}\bar{z}_2\bar{\xi}_1 - \sqrt{\frac{\mu_1}{\mu_2}}\bar{z}_1\bar{\xi}_2 \right) \cdot \left( \sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^* \right).$$

Now, by Corollary 3.4, we have

$$\phi_{33}^{(4,0)} = \frac{2}{\sqrt{\mu_1 + \mu_2}} \left( \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^* - \sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* \right). \quad (4.37)$$

Moreover

$$2i\left(\frac{\overline{\xi_1}\eta_1^*}{\mu_1} + \frac{\overline{\xi_2}\eta_2^*}{\mu_2}\right) = 2i|z|^2\overline{z}f^{(2,2)}(z) + \overline{\Phi_0^{(2,0)}}(z) \cdot \Phi_0^{(3,1)}(z). \quad (4.38)$$

Write

$$\widetilde{\phi}^{(3,1)}(z) = \phi^{(3,1)}(z) - 2i\left(\frac{\eta_1^*}{\mu_1}e_1^* + \frac{\eta_2^*}{\mu_2}e_2^*\right), \quad \widetilde{\Phi}_0^{(3,1)}(z) = \Phi_0^{(3,1)}(z) - 2i\left(\frac{\eta_1}{\mu_1}e_1 + \frac{\eta_2}{\mu_2}e_2\right). \quad (4.39)$$

Then we have

$$\overline{\Phi_0^{(2,0)}}(z)\widetilde{\Phi}_0^{(3,1)}(z) = -2i|z|^2\overline{z}f^{(2,2)}(z).$$

Hence, we get

$$\begin{aligned} \mu_{11} \cdot \widetilde{\Phi}_{11}^{(3,1)} &= -2iz_1f_1^{(2,2)}, \\ \mu_{12} \cdot \widetilde{\Phi}_{12}^{(3,1)} &= -2i(z_1f_2^{(2,2)} + z_2f_1^{(2,2)}), \\ \mu_{22} \cdot \widetilde{\Phi}_{22}^{(3,1)} &= -2iz_2f_2^{(2,2)}, \\ \mu_{1j} \cdot \widetilde{\Phi}_{1j}^{(3,1)} &= -2iz_jf_1^{(2,2)}, \quad j \geq 3 \\ \mu_{2j} \cdot \widetilde{\Phi}_{2j}^{(3,1)} &= -2iz_jf_2^{(2,2)} \quad j \geq 3. \end{aligned} \quad (4.40)$$

By Lemma 3.1, we have

$$\begin{aligned} \overline{\Phi_0^{(3,0)}}\widetilde{\Phi}_0^{(3,1)} &= -4|z|^2\left(\frac{\overline{\xi_1}}{\mu_1}f_1^{(2,2)} + \frac{\overline{\xi_2}}{\mu_2}f_2^{(2,2)}\right) \\ &\quad + \frac{4}{\mu_1 + \mu_2}\left(\sqrt{\frac{\mu_1}{\mu_2}}\overline{z_1\xi_2} - \sqrt{\frac{\mu_2}{\mu_1}}\overline{z_2\xi_1}\right) \cdot \left(\sqrt{\frac{\mu_1}{\mu_2}}z_1f_2^{(2,2)} - \sqrt{\frac{\mu_2}{\mu_1}}z_2f_1^{(2,2)}\right). \end{aligned} \quad (4.41)$$

Notice that

$$\overline{\Phi_0^{(3,0)}} \cdot 2i\left(\frac{\eta_1^*}{\mu_1}e_1 + \frac{\eta_2^*}{\mu_2}e_2\right) = 2i\left(\frac{\eta_1^*}{\mu_1}\overline{\eta_1} + \frac{\eta_2^*}{\mu_2}\overline{\eta_2}\right). \quad (4.42)$$

Hence

$$\begin{aligned} \overline{\Phi_0^{(3,0)}}\Phi_0^{(3,1)} &= 2i\left(\frac{\eta_1^*}{\mu_1}\overline{\eta_1} + \frac{\eta_2^*}{\mu_2}\overline{\eta_2}\right) - 4|z|^2\left(\frac{\overline{\xi_1}}{\mu_1}f_1^{(2,2)} + \frac{\overline{\xi_2}}{\mu_2}f_2^{(2,2)}\right) \\ &\quad + \frac{4}{\mu_1 + \mu_2}\left(\sqrt{\frac{\mu_1}{\mu_2}}\overline{z_1\xi_2} - \sqrt{\frac{\mu_2}{\mu_1}}\overline{z_2\xi_1}\right) \cdot \left(\sqrt{\frac{\mu_1}{\mu_2}}z_1f_2^{(2,2)} - \sqrt{\frac{\mu_2}{\mu_1}}z_2f_1^{(2,2)}\right). \end{aligned} \quad (4.43)$$

Combining (4.33) with (4.36) and making use of Lemma 2.2 and Corollary 3.4 again, we get

$$4|z|^2 \overline{\Phi_0^{(1,2)}(z)} \cdot \Phi_0^{(2,0)}(z) + 2i \overline{\phi^{(2,1)}(z)} \cdot \phi^{(3,0)}(z) = 4\left(\frac{1}{\mu_1} \overline{\xi_1} \eta_1^* + \frac{1}{\mu_2} \overline{\xi_2} \eta_2^*\right) \quad (4.44)$$

Namely, we have

$$|z|^2 A(z, \bar{z}) + 2i \overline{\widetilde{\phi}^{(2,1)}(z)} \cdot \phi^{(3,0)}(z) = 0. \quad (4.45)$$

Here, as before, we write  $A(z, \bar{z})$  for a real analytic function which may be different in different contexts.

Combining (4.24) with (4.45) and making use of Lemma 2.2, we get

$$\widetilde{\phi}_{33}^{(2,1)}(z) = \frac{-2}{\sqrt{\mu_1 + \mu_2}} \left( \sqrt{\frac{\mu_1}{\mu_2}} z_1 f_2^{(1,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}} z_2 f_1^{(1,2)}(z) \right). \quad (4.46)$$

Next we will prove that  $\phi_{3j}^{(4,0)} = 0$ ,  $\widetilde{\phi}_{3j}^{(2,1)}(z) = 0$  for  $j = 4, \dots, K$  with  $K = N - n - (n - 1) - (n - 2)$ .

Considering terms of weighted degree 8 in the basic equation (3.3), we get

$$2\operatorname{Re}\left\{\bar{z}f^{(7)} + \overline{f^{(3)}}f^{(5)} + \overline{\Phi_0^{(2)}}\Phi_0^{(6)} + \overline{\phi^{(3)}}\phi^{(5)}\right\} + |f^{(4)}|^2 + |\phi^{(4)}|^2 = 0. \quad (4.47)$$

Namely, we have

$$\begin{aligned} & 2\operatorname{Re}\left\{\bar{z}\left(f^{(5,1)}(u + i|z|^2) + f^{(3,2)}(u + i|z|^2)^2 + f^{(1,3)}(u + i|z|^2)^3\right)\right. \\ & + \overline{f^{(1,1)}(u + i|z|^2)} \cdot \left(f^{(3,1)}(u + i|z|^2) + f^{(1,2)}(u + i|z|^2)^2\right) \\ & + \overline{\Phi_0^{(2,0)}} \cdot \left(\Phi_0^{(6,0)} + \Phi_0^{(4,1)}(u + i|z|^2) + \Phi_0^{(2,2)}(u + i|z|^2)^2\right) \\ & + \left(\overline{\phi^{(3,0)}} + \overline{\phi^{(1,1)}(u + i|z|^2)}\right) \cdot \left(\phi^{(5,0)} + \phi^{(3,1)}(u + i|z|^2) + \phi^{(1,2)}(u + i|z|^2)^2\right)\Big\} \\ & + \left|f^{(2,1)}(u + i|z|^2)\right|^2 + \left|\phi^{(4,0)} + \phi^{(2,1)}(u + i|z|^2)\right|^2 = 0. \end{aligned} \quad (4.48)$$

Collecting terms of the form  $z^\alpha \bar{z}^\beta$  with  $|\alpha| = 4, |\beta| = 4$ , we get

$$\begin{aligned} & 2\operatorname{Re}\left\{-i\bar{z}f^{(1,3)}|z|^6 - \overline{f^{(1,1)}}(-i|z|^2)f^{(1,2)}|z|^4 + \overline{\Phi_0^{(2,0)}}\Phi_0^{(2,2)}(-|z|^4)\right. \\ & + \overline{\phi^{(3,0)}}\phi^{(3,1)}i|z|^2 + \overline{\phi^{(1,1)}}(-i|z|^2)\phi^{(1,2)}(-|z|^4)\Big\} \\ & + \left|\phi^{(4,0)}\right|^2 + (|f^{(2,1)}|^2 + |\phi^{(2,1)}|^2)|z|^4 = 0. \end{aligned} \quad (4.49)$$



Collecting terms of the form  $z^\alpha \bar{z}^\beta u^2$  with  $|\alpha| = 2, |\beta| = 2$ , we get

$$2\operatorname{Re}\left\{\bar{z}f^{(1,3)}3i|z|^2 + \overline{f^{(1,1)}}f^{(1,2)}i|z|^2 + \overline{\Phi_0^{(2,0)}}\Phi_0^{(2,2)} + \overline{\phi^{(1,1)}}\phi^{(1,2)}i|z|^2\right\} \\ + (|f^{(2,1)}|^2 + |\phi^{(2,1)}|^2) = 0. \quad (4.50)$$

Collecting terms of the form  $z^\alpha \bar{z}^\beta u$  with  $|\alpha| = 3, |\beta| = 3$ , we get

$$2\operatorname{Re}\left\{\bar{z}f^{(1,3)}3(-|z|^4) + \overline{f^{(1,1)}}f^{(1,2)}i|z|^4 + \overline{\Phi_0^{(2,0)}}\Phi_0^{(2,2)}2i|z|^2 \right. \\ \left. + \overline{\phi^{(3,0)}}\phi^{(3,1)} + \overline{\phi^{(1,1)}}\phi^{(1,2)}i|z|^4\right\} = 0. \quad (4.51)$$

By (4.49) and (4.50), we get

$$|z|^2 \cdot 2\operatorname{Re}\left\{-4i\bar{z}f^{(1,3)}|z|^4 - 2\overline{\Phi_0^{(2,0)}}\Phi_0^{(2,2)}(|z|^2) + i\overline{\phi^{(3,0)}}\phi^{(3,1)}\right\} + \left|\phi^{(4,0)}\right|^2 = 0. \quad (4.52)$$

Combining this with (4.51), we get

$$|z|^6 A(z, \bar{z}) + 2|z|^2 \cdot \left(-2\overline{\Phi_0^{(2,0)}}\Phi_0^{(2,2)}(|z|^2) + i\overline{\phi^{(3,0)}}\phi^{(3,1)}\right) + \left|\phi^{(4,0)}\right|^2 = 0. \quad (4.53)$$

By (4.35) and Lemma 3.1, we get

$$\frac{1}{4}|\Phi_0^{(4,0)}|^2 = |z|^2 \left(\frac{1}{\mu_1}|\eta_1^*|^2 + \frac{1}{\mu_2}|\eta_2^*|^2\right) - \frac{1}{\mu_1 + \mu_2} \left|\sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^*\right|^2. \quad (4.54)$$

Combining this with (4.53) and making use of Lemma 2.2, we get

$$|z|^4 A(z, \bar{z}) - 4|z|^2 \overline{\Phi_0^{(2,0)}}\Phi_0^{(2,2)} + 2i\overline{\phi^{(3,0)}}\phi^{(3,1)} + \frac{4}{\mu_1}|\eta_1^*|^2 + \frac{4}{\mu_2}|\eta_2^*|^2 = 0. \quad (4.55)$$

and

$$\frac{1}{4}|\Phi_1^{(4,0)}|^2 = \frac{1}{\mu_1 + \mu_2} \left|\sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^* - \sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^*\right|^2. \quad (4.56)$$

By (4.37) and (4.56), we get

$$\phi_{33}^{(4,0)} = \frac{2}{\sqrt{\mu_1 + \mu_2}} \left(\sqrt{\frac{\mu_1}{\mu_2}}z_1\eta_2^* - \sqrt{\frac{\mu_2}{\mu_1}}z_2\eta_1^*\right) \\ \phi_{3j}^{(4,0)} = 0 \text{ for } j > 3. \quad (4.57)$$

This proves Theorem 4.1 (1).

Substituting (4.43) into (4.55), we get

$$\begin{aligned}
& |z|^4 A(z, \bar{z}) - 4|z|^2 \overline{\Phi_0^{(2,0)}} \Phi_0^{(2,2)} + 2i \overline{\Phi_1^{(3,0)}} \Phi_1^{(3,1)} - 8i|z|^2 \left( \frac{\bar{\xi}_1}{\mu_1} f_1^{(2,2)} + \frac{\bar{\xi}_2}{\mu_2} f_2^{(2,2)} \right) \\
& + \frac{8i}{\mu_1 + \mu_2} \left( \sqrt{\frac{\mu_1}{\mu_2}} \bar{z}_1 \bar{\xi}_2 - \sqrt{\frac{\mu_2}{\mu_1}} \bar{z}_2 \bar{\xi}_1 \right) \cdot \left( \sqrt{\frac{\mu_1}{\mu_2}} z_1 f_2^{(2,2)} - \sqrt{\frac{\mu_2}{\mu_1}} z_2 f_1^{(2,2)} \right) \\
& + \frac{4}{\mu_1} \eta_1^* (\eta_1^* - \eta_1) + \frac{4}{\mu_2} \eta_2^* (\eta_2^* - \eta_2) = 0.
\end{aligned} \tag{4.58}$$

Notice that  $\Phi_1^{(3,0)} = (\phi_{33}^{(3,0)}, 0, \dots, 0)$  and  $n \geq 7$ . Making use of Lemma 2.2, we get

$$\overline{\Phi_0^{(2,0)}} \Phi_0^{(2,2)} = -2i \left( \frac{\bar{\xi}_1}{\mu_1} f_1^{(2,2)} + \frac{\bar{\xi}_2}{\mu_2} f_2^{(2,2)} \right) + |z|^2 A(z, \bar{z}). \tag{4.59}$$

By (4.29), we have

$$\begin{aligned}
f_1^{(2,2)} &= \frac{i}{2} \mu_1 f_1^{(2,1)} - \overline{\Phi_0^{(I_1+2I_n)}} \Phi_0^{(2,0)} - \bar{e}_1^* \phi^{(2,1)}, \\
f_2^{(2,2)} &= \frac{i}{2} \mu_2 f_2^{(2,1)} - \overline{\Phi_0^{(I_2+2I_n)}} \Phi_0^{(2,0)} - \bar{e}_2^* \phi^{(2,1)}.
\end{aligned} \tag{4.60}$$

Thus we get

$$2\text{Re} \left( -2i \left( \frac{\bar{\xi}_1}{\mu_1} f_1^{(2,2)} + \frac{\bar{\xi}_2}{\mu_2} f_2^{(2,2)} \right) \right) = I + II + III. \tag{4.61}$$

Here

$$\begin{aligned}
I &= 2\text{Re} \left\{ -2i \left( \frac{\bar{\xi}_1}{\mu_1} \frac{i}{2} \mu_1 (-\xi_1) + \frac{\bar{\xi}_2}{\mu_2} \frac{i}{2} \mu_2 (-\xi_2) \right) \right\} = -2(|\xi_1|^2 + |\xi_2|^2). \\
II &= 2\text{Re} \left( 2i \frac{\bar{\xi}_1}{\mu_1} \overline{\Phi_0^{(I_1+2I_n)}} \Phi_0^{(2,0)} + 2i \frac{\bar{\xi}_2}{\mu_2} \overline{\Phi_0^{(I_2+2I_n)}} \Phi_0^{(2,0)} \right) \\
&= (|\xi_1|^2 + |\xi_2|^2) + 4 \left| \frac{\xi_1}{\mu_1} e_1^* + \frac{\xi_2}{\mu_2} e_2^* \right|^2. \\
III &= 2\text{Re} \left( 2i \frac{\bar{\xi}_1}{\mu_1} \bar{e}_1^* \phi^{(2,1)} + 2i \frac{\bar{\xi}_2}{\mu_2} \bar{e}_2^* \phi^{(2,1)} \right).
\end{aligned} \tag{4.62}$$

The equality for  $I$  follows from (3.5) and  $II$  follows from (4.26). By (4.50), we get

$$|z|^2 A(z, \bar{z}) + 2\text{Re} \left\{ \overline{\Phi_0^{(2,0)}} \Phi_0^{(2,2)} \right\} + (|f^{(2,1)}|^2 + |\phi^{(2,1)}|^2) = 0. \tag{4.63}$$

Substituting (4.61) and (4.62) into (4.63), we get

$$\begin{aligned}
& |z|^2 A(z, \bar{z}) - 2(|\xi_1|^2 + |\xi_2|^2) + (|\xi_1|^2 + |\xi_2|^2) + 4\left|\frac{\xi_1}{\mu_1}e_1^* + \frac{\xi_2}{\mu_2}e_2^*\right|^2 \\
& + 2\operatorname{Re}\left(2i\frac{\bar{\xi}_1}{\mu_1}\bar{e}_1^*\phi^{(2,1)} + 2i\frac{\bar{\xi}_2}{\mu_2}\bar{e}_2^*\phi^{(2,1)}\right) + (|\xi_1|^2 + |\xi_2|^2) + |\phi^{(2,1)}|^2 = 0.
\end{aligned} \tag{4.64}$$

Hence we get

$$|z|^2 A(z, \bar{z}) + \left|\phi^{(2,1)}(z) - 2i\left(\frac{\xi_1}{\mu_1}e_1^* + \frac{\xi_2}{\mu_2}e_2^*\right)\right|^2 = 0. \tag{4.65}$$

Substituting (4.23) into (4.65), we get

$$|z|^2 A(z, \bar{z}) + \left|\tilde{\Phi}_1^{(2,1)}\right|^2 - \frac{4}{\mu_1 + \mu_2} \left|\sqrt{\frac{\mu_1}{\mu_2}}z_1 f_2^{(1,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}}z_2 f_1^{(1,2)}(z)\right|^2 = 0. \tag{4.66}$$

Making use of (4.46) and Lemma 2.2, we get

$$\begin{aligned}
\tilde{\Phi}_{33}^{(2,1)} &= \frac{-2}{\sqrt{\mu_1 + \mu_2}} \left( \sqrt{\frac{\mu_1}{\mu_2}}z_1 f_2^{(1,2)}(z) - \sqrt{\frac{\mu_2}{\mu_1}}z_2 f_1^{(1,2)}(z) \right) \\
\tilde{\Phi}_{3j}^{(2,1)} &= 0 \text{ for } j > 3.
\end{aligned} \tag{4.67}$$

The proof of Theorem 4.1 (2) is also complete.

## 5 Proofs of Theorem 1.2 and Theorem 1.3

**Step (I): An immediate application of a normal form in [HJX1] for maps with geometric rank 1:** We first consider  $F \in \operatorname{Rat}(\mathbb{B}^n, \mathbb{B}^N)$  with geometric rank 1. Then by Theorem 1.2 of [HJX1],  $F$  is equivalent to a map of the form  $\Phi = (z_1, \dots, z_{n-1}, z_n H(z)) := (\phi_1, \dots, \phi_N)$  with  $H \in \operatorname{Rat}(\mathbb{B}^n, \mathbb{B}^{N-n+1})$  also of geometric rank one. We notice that linear fractional transformations map affine linear subspaces to affine linear subspaces. Also,  $F(\mathbb{B}^n)$  is contained in an affine subspace of dimension  $m$ , if and only if  $F$  is equivalent to a map of the form  $(G, 0)$  with  $G$  having  $m$ -components. Now suppose the image of  $H = (h_1, \dots, h_{N-n+1})$  is contained in an affine subspace of dimension  $m < N - n$ , then there are  $(N - n - m + 1)$  linearly independent vectors  $\mu_j = (a_{j1}, \dots, a_{jk})$  with  $k = N - n + 1$  such that  $\sum_{l=1}^k a_{jl} h_l(z) \equiv c_l$  for certain  $c_l \in \mathbb{C}$ . If  $c_l = 0$  for all  $l$ , then  $\sum_{l=1}^k a_{jl} \phi_{n-1+l}(z) \equiv 0$ . Hence  $\Phi(\mathbb{B}^n)$  is contained in an affine linear subspace of dimension  $m + n - 1$ . Otherwise, assume without of generality that  $c_1 = 1$ . Then we have  $\sum_{l=1}^k (a_{jl} - c_l a_{1l}) \phi_{n-1+l}(z) \equiv 0$ . Notice that  $\{\mu_2 - c_2 \mu_1, \dots, \mu_{N-n-m+1} -$

$c_{N-n-m+1}\mu_1\}$  is also linearly independent, we see that  $\Phi(\mathbb{B}^n)$  is contained in an affine subspace of dimension  $m+n$ . Now, starting with  $N=3n-4$  and applying the gap rigidity in [HJX1] to  $H$ , we can see that when  $N \leq 4n-7$ ,  $F(\mathbb{B}^n)$  is contained in an affine linear subspace of dimension  $3n$ . By an induction argument, we see that when  $N < (k+1)n - \frac{k(k+1)}{2}$ ,  $F(\mathbb{B}^n)$  is contained in a linear affine subspace of dimension  $kn$ , if  $(k+1)n - \frac{k(k+1)}{2} > 0$ . In particular, when  $N \leq 4n-7$  and  $F$  has geometric rank 1, then  $F$  is equivalent to a map of the form  $(G, 0)$  with  $G \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{3n})$ . This proves Theorem 1.2 in case the map  $F$  has geometric rank one.

**Step (II): Completion of the Proofs of Theorems 1.2 and 1.3:** By Lemma 3.2 in [Hu2], when  $N \leq 4n-7$ , any  $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$  can only have geometric rank  $\kappa_0 = 0, 1$ , or 2. When  $\kappa_0 = 0$ ,  $F$  is linear and thus Theorem 1.2 follows trivially. When  $\kappa_0 = 1$ , the proof of Theorem 1.2 is already done in Step (I). The case of Theorem 1.2 for maps with geometric rank two is a special case of Theorem 1.3. Hence, it suffices now to prove Theorem 1.3.

Let  $\kappa_0 = 2$  be such that  $N \leq 4n-6$ . Let  $F \in \text{Prop}_3(\mathbb{B}^n, \mathbb{B}^N)$  with geometric rank  $\kappa_0 = 2$ . As mentioned in §2, we can assume, without loss of generality, that  $F \in \text{Rat}(\mathbf{H}^n, \mathbf{H}^N)$  satisfies the normalization in Theorem 2.1. By Theorem 4.1 (1), (4.35) and the normalization of  $F$  at 0, it follows that for any  $\alpha$  with  $|\alpha| = 4$ ,  $\mathcal{L}^\alpha F|_0 \in \text{span}_{|\beta| \leq 3} \{\mathcal{L}^\beta F|_0\}$ . Here  $\mathcal{L}_j = \frac{\partial}{\partial z_j} - 2i\bar{z}_j \frac{\partial}{\partial w}$

is a tangent vector field of type  $(1, 0)$  along  $\partial\mathbf{H}^n$  for  $j = 1, \dots, n-1$ , and  $\mathcal{L}^\alpha$  is defined in the standard way. Also notice that at the origin  $\mathcal{L}^\alpha = D_Z^\alpha$  with  $D$  the regular differentiation operator. Now, we proceed in the same way as in [Hu1], though the situation in [Hu1] is harder for the maps there are only assumed to be twice differentiable. We first assume that  $\phi_{33}^{(3,0)} \neq 0$ .

For any  $p \in \mathbf{H}^n$  ( $\approx 0$ ), there exists  $\tau \in \text{Aut}_0(\mathbf{H}^N)$ ,  $\sigma \in \text{Aut}_0(\mathbf{H}^n)$  such that  $G = \tau \circ F_p \circ \sigma$  satisfies the normalization condition as in Theorem 2.1. For any  $|\alpha| = 4$ , it can be easily verified that

$$\mathcal{L}^\alpha(\tau^{-1} \circ G \circ \sigma^{-1})|_0 \in \text{span}_{|\beta| \leq 3} \{\mathcal{L}^\beta(\tau^{-1} \circ G \circ \sigma^{-1})|_0\}. \quad (5.1)$$

This immediately gives  $\mathcal{L}^\alpha F(p) \in \text{span}_{|\beta| \leq 3} \{\mathcal{L}^\beta F(p)\}$ . Since  $\text{span}_{|\beta| \leq 3} \{\mathcal{L}^\beta F(p)\}$  has a fixed dimension

for  $p \approx 0$ , we can write, for any  $\alpha$ ,  $\mathcal{L}^\alpha F(p)$  as a smooth linear combination of a fixed basis from  $\text{span}_{|\beta| \leq 3} \{\mathcal{L}^\beta F(p)\}$ . Successively applying  $\bar{\mathcal{L}}_j$ ,  $\mathcal{L}_k$  as in [Hu1] to the so obtained expressions

and using the bracket property for such vector fields, we can obtain as in [Hu1] that  $D^\alpha F(0) \in \text{span}_{|\beta| \leq 3} \{D^\beta F(0)\}$  for any multiple index  $\alpha$ . Here  $D^\alpha$  is the regular differentiation of order  $|\alpha|$ .

Thus  $F(z, w) \in \text{span}_{|\beta| \leq 3} \{D^\beta F(0)\}$  by the Taylor expansion for  $(z, w) \approx 0$ . Now, write as before,

$\phi^{(1,1)}(z)w = (e_1^*z_1 + e_2^*z_2)w$ . By Theorem 4.1 (2) (3), we see that  $\text{span}\{D^\beta F(0)\}_{\beta \leq 3}$  stays in the

span of the following vectors:  $\left\{ (0, \dots, 0, 1^{j^{th}}, 0, \dots, 0), (0, \dots, 0, 1), (0, 0, \dots, 0, \hat{e}_1, 0), (0, \dots, 0, \hat{e}_2, 0) \right\}$ , where  $1 \leq j \leq ((n-1) + (n-1) + (n-2)) + 1$ . Hence  $F(\mathbb{B}^n)$  is contained in a linear subspace with dimension equal to  $3n - 3 + 2 + 1 = 3n$ . Back to the balls by Cayley transformations, we complete the proof of Theorem 1.3 and thus also the proof of Theorem 1.2 in this setting. Now, assume that for a certain  $p_0 \approx 0$ , the  $(\phi)_{33}^{(3,0)} \neq 0$  associated with  $F_{p_0}$ , then we can consider  $F_{p_0}$  instead of  $F$  and apply the above argument to conclude the proof of Theorem 1.2. Finally, if after the normalization of  $F_p$  to the form as in Theorem 1.3 for any  $p \approx 0$ , we have  $\phi^{(3,0)} \equiv 0$ , then a similar method as above shows that  $\mathcal{L}^\alpha F(p) \in \text{span}\{\mathcal{L}^\beta F(0)\}_{|\beta| \leq 2}$ . Hence,  $F(\mathbb{B}^n)$  is contained in a complex linear subspace of dimension

$n + (n-1) + (n-2) + 2 = 3n - 1$ , spanned by  $\left\{ (0, \dots, 0, 1^{j^{th}}, 0, \dots, 0), (0, \dots, 0, 1), (0, 0, \dots, 0, \hat{e}_1, 0), (0, \dots, 0, \hat{e}_2, 0) \right\}$ , where  $1 \leq j \leq ((n-1) + (n-1) + (n-2))$ .

□

**Remark:** Consider the map defined in (1.1):

$$F = (z_1, \dots, z_{n-2}, \lambda z_{n-1}, z_n, \sqrt{1 - \lambda^2} z_{n-1}(z_1, \dots, z_{n-1}, \mu z_n, \sqrt{1 - \mu^2} z_n z)), \quad \lambda, \mu \in (0, 1).$$

The map  $F$  is of degree three with geometric rank two and can be easily seen not to equivalent to a map of the form  $(G, 0)$ . In fact, if  $\phi_{33}^{(3,0)} \equiv 0$  for such a map in a generic situation, namely, if it holds even for the normalization to the form as in Theorem 1.3 for  $F_p$  for a generic  $p$ , then we see that  $F(\mathbb{B}^n)$  is contained in  $\mathbb{C}^{3n-1}$ . But this is impossible. Hence, for such a map and at a generic point, the third order derivative does not stay in the span of the lower order derivatives. (We also refer the reader to the paper of Lamel-Mir [LM] and many references therein for other related jet determination and parametrization problems.)

**Acknowledgement:** This paper is a simplified version of the authors' early preprint. (Theorem 1.2 was first announced in [HJY] (Theorem 2.9 in [HJY])). The authors would like to thank P. Ebenfelt, N. Mir and D. Zaitsev for their helpful conversations related to this work.

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